Midterm Exam Calculus 2

17 March 2017, 9:00-11:00

university of groningen

The midterm exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[10+10+5 Points]** Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine for which unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ the directional derivative $D_{\boldsymbol{u}}f(0,0)$ exist.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. **[10+10 Points]** Consider the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = (\sin t - t\cos t)\,\mathbf{i} + 2\,\mathbf{j} + (\cos t + t\sin t)\,\mathbf{k}.$$

   (a) Determine the parametrization by arc length.

   (b) At each point on the curve, determine the curvature of the curve.

3. **[5+10+10 Points]** Consider the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$

   which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Determine the tangent plane of the ellipsoid at the point (x_0, y_0, z_0).

   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto z$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

   (c) Use the method of Lagrange multipliers to find the points closest to and farthest away from the origin.

4. **[20 Points]** Determine

$$\iiint_W (2 + \sqrt{x^2+y^2})\,\mathrm{d}V,$$

   where

$$W = \{(x,y,z) \in \mathbb{R}^3 | \sqrt{x^2+y^2} \leq \frac{z}{2} \leq 3\}.$$

1. a) use polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \frac{x^2y^2}{x^4+y^4} = \frac{r^4\cos^2\theta\sin^2\theta}{r^4(\cos^4\theta+\sin^4\theta)} = \frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}$$

For $r\to 0$, the latter expression has a limit that depends on θ. For example for $\theta = \frac{\pi}{4}$ the limit is

$$\frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}$$

which does not equal $f(0,0)=0$.

b) Let $u = (v,w)\in\mathbb{R}^2$ unit vector, i.e. $v^2+w^2=1$.
Then for the directional derivative $D_uf(0,0)$ to exist the difference quotient

$$\frac{f(tv,tw)-f(0,0)}{t}$$

must have a limit for $t\to 0$.

We have for $t\neq 0$,

$$\frac{f(tv,tw)-f(0,0)}{t} = \frac{\frac{t^4v^2w^2}{t^4(v^4+w^4)}-0}{t}$$

$$= \frac{1}{t}\,\frac{v^2w^2}{v^4+w^4}.$$ This has a limit for $t\to 0$

only if $v^2w^2=0$, i.e. $v=0$ or $w=0$.

This means that only the directional derivatives in the direction of the x-axis and the y-axis exist.

The derivatives are $f_x(0,0) = f_y(0,0) = 0$.

c) f is not differentiable at $(0,0)$ because f is not continuous at $(0,0)$.

Also if f was differentiable at $(0,0)$ then the directional derivative in part (b) would equal $u \cdot \nabla f(0,0) = v f_x(0,0) + w f_x(0,0) = 0$.

But the directional derivative does in general not exist as we saw in part (b).

2. a) $r'(t) = (\cos t - \cos t + t\sin t)\,i + (-\sin t + \sin t + t\cos t)\,k$

$= t\sin t\, i + t\cos t\, k$

$\Rightarrow \|r'(t)\| = [\,t^2\sin^2 t + t^2\cos^2 t\,]^{1/2} = t$

arclength $s(t) = \int_0^t \|r'(\tau)\|\,d\tau = \int_0^t \tau\,d\tau = \frac{1}{2}t^2$

$\Rightarrow t(s) = (2s)^{1/2}$

parametrization by arclength

$\tilde{r}(s) = r(t(s)) = \left(\sin(2s)^{1/2} - (2s)^{1/2}\cos(2s)^{1/2}\right) i + 2j + \left(\cos(2s)^{1/2} + (2s)^{1/2}\sin(2s)^{1/2}\right) k$

where $s \in [s(0), s(2\pi)] = [0, \frac{1}{2}(2\pi)^2] = [0, 2\pi^2]$

b) curvature at $r(t)$:

$K = \frac{1}{\|r'(t)\|}\left\|\frac{dT}{dt}\right\|$ where T is the unit tangent vector at $r(t)$ which is

$T = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{t}(t\sin t\, i + t\cos t\, k)$

$= \sin t\, i + \cos t\, k$

$\Rightarrow K = \frac{1}{t}\left\|\frac{d}{dt}(\sin t\, i + \cos t\, k)\right\| = \frac{1}{t}\|\cos t\, i - \sin t\, k\| = \frac{1}{t}$

$t \in [0, 2\pi]$

3. a) Set $F(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$

$\Rightarrow x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \Leftrightarrow F(x,y,z) = 3$

Then $\nabla F(x_0, y_0, z_0)$ is perpendicular to the ellipsoid at (x_0, y_0, z_0).

As $\nabla F(x,y,z) = 2x\,i + \frac{1}{2}y\,j + \frac{2}{9}z\,k$

we get $\nabla F(x_0,y_0,z_0) = \nabla F(1,2,3) = 2i + 1j + \frac{2}{3}k$

$\Rightarrow$ tangent plane is given by the equation

$$\nabla F(x_0,y_0,z_0)(r - r_0) = 0$$

where $r = xi + yj + zk$

$r_0 = x_0 i + y_0 j + z_0 k$

which gives

$$\left(2i + 1j + \frac{2}{3}k\right) \cdot \left((x-1)i + (y-2)j + (z-3)k\right) = 0$$

$\Leftrightarrow \quad (2x-2) + (y-2) + \frac{2}{3}(z-3) = 0$

$\Leftrightarrow \quad 2x + y + \frac{2}{3}z = 6$

b) For F as defined in part (a) we have

$$\frac{\partial}{\partial z} F(x,y,z) = \frac{2}{9} z \text{ which gives } \frac{\partial}{\partial z} F(x_0,y_0,z_0) = \frac{2}{9} z_0 = \frac{2}{3}.$$

As $\frac{\partial}{\partial z} F(x_0,y_0,z_0) \neq 0$ there exist by the Implicit Function Theorem a neighborhood U of (x_0,y_0) in $\mathbb{R}^2$, a neighb. V of z_0 in $\mathbb{R}$ and a function $f: U \to V$ such that if $F(x,y,z) = 3$ for $(x,y) \in U$ and $z \in V$ then $z = f(x,y)$. Both F and f are C^1 functions

$$f_x(x_0,y_0) = -\left.\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=(x_0,y_0,z_0)} = -\frac{2x_0}{\frac{2}{9}z_0} = -9\frac{x_0}{z_0} = -3$$

$$f_y(x_0,y_0) = -\left.\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=(x_0,y_0,z_0)} = -\frac{\frac{1}{2}y_0}{\frac{2}{9}z_0} = -\frac{9}{4}\frac{2}{3} = -\frac{3}{2}$$

Linearization of f at (x_0,y_0):

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 3 - 3(x-1) - \frac{3}{2}(y-2)$$

The graph of L satisfies

$$z = L(x,y) \iff z = 3 - 3(x-1) - \frac{3}{2}(y-2)$$

$$\iff z = 3 - 3x + 3 - \frac{3}{2}y + 3$$

$$\iff z = 9 - 3x - \frac{3}{2}y$$

$$\iff \frac{2}{3}z + 2x + y = 6 \quad \text{(agrees with (a))}$$

3. c) distance square of the point (x,y,z) to the origin is given by $g(x,y,z) = x^2+y^2+z^2$.

Let F be defined as the part (a). Then at an extremum of g restricted to $F(x,y,z)=3$ there exist $\lambda \in \mathbb{R}$ such that $\nabla g(x,y,z) = \lambda \nabla F(x,y,z)$.

We have to solve the latter equation together with $F(x,y,z)=3$ for x,y,z and λ.

$$\left.\begin{aligned} 2x &= \lambda 2x \\ 2y &= \lambda \tfrac{1}{2} y \\ 2z &= \lambda \tfrac{2}{9} z \\ x^2 + \tfrac{y^2}{4} + \tfrac{z^2}{9} &= 3 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} &x=0 \text{ or } \lambda=1 \\ &y=0 \text{ or } \lambda=4 \\ &z=0 \text{ or } \lambda=9 \\ &x^2+\tfrac{y^2}{4}+\tfrac{z^2}{9}=3 \end{aligned}\right\}$$

$\Leftrightarrow$ $x=0,\ y=0,\ \lambda=9$

or

$x=0,\ z=0,\ \lambda=4$

or

$y=0,\ z=0,\ \lambda=1$

with $x^2+\frac{y^2}{4}+\frac{z^2}{9}=3$

$\Leftrightarrow$ $z^2=27,\ \lambda=9$

or

$y^2=12,\ \lambda=4$

or

$x^2=3,\ \lambda=1$

Filling in the resulting points into g gives

$g(0,0,\pm\sqrt{27}) = 27$

$g(0,\pm\sqrt{12},0) = 12$

$g(\pm\sqrt{3},0,0) = 3$

$\Rightarrow$ at $(x,y,z)=(0,0,\pm\sqrt{27})$ points on the ellipsoid are furthest away and at $(x,y,z)=(\pm\sqrt{3},0,0)$ points on the ellipsoid are closest to the origin

4. Cylinder coordinates $\qquad W = \{(r,\theta,z) \mid r \le \frac{z}{2} \le 3\}$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\int_0^3 \int_0^{2\pi} \int_{2r}^{6} (2+r)\, r \, dz\, d\theta\, dr$$

$$= 2\pi \int_0^3 (2+r)\, r \cdot (6-2r)\, dr$$

$$= 2\pi \int_0^3 r(12 - 4r + 6r - 2r^2)\, dr$$

$$= 2\pi \int_0^3 r(12 + 2r - 2r^2)\, dr$$

$$= 2\pi \left(6r^2 + \frac{2}{3}r^3 - \frac{1}{2}r^4\right)\Bigg|_{r=0}^{r=3}$$

$$= 2\pi \left(6\cdot 9 + 2\cdot 9 - \frac{81}{2}\right)$$

$$= 2\pi \left(72 - \frac{81}{2}\right)$$

$$= 63\pi$$